HOW TO Develop Problem Solving Using a Calculator

Janet Morris

National Council of Teachers of Mathematics

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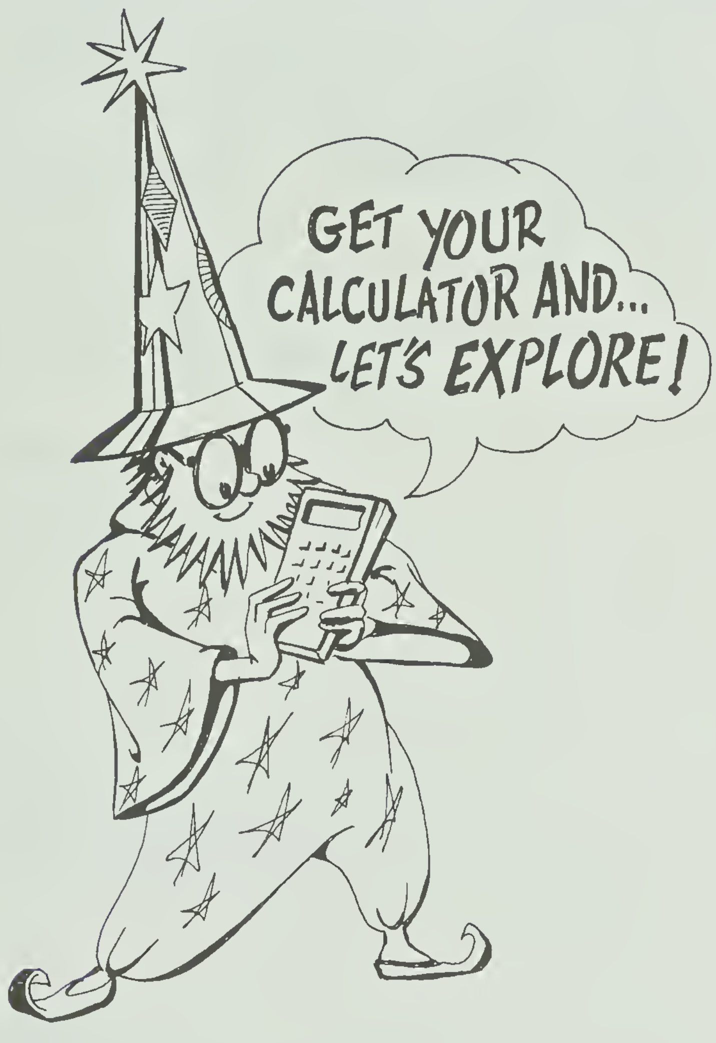

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Second printing 1983
Third printing 1985

Library of Congress Cataloging in Publication Data:

Morris, Janet.
How to develop problem solving using a calculator.

1. Problem solving—Data processing. 2. Mathematics—
Problems, exercises, etc.—Data processing.
3. Calculating-machines. I. Title.
QA63.M65 513'.028'54 81-9569
ISBN 0-87353-175-2 AACR2

Printed in the United States of America

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PREFACE

This collection of activities illustrates how you can help students develop problem-solving techniques or strategies using a basic four-function calculator. The techniques, *look for a pattern, make a chart* or *organized list,* and *guess and check,* are used repeatedly in activities that range from discovery to application to strategy games. The repeated use of the strategies in a variety of settings teaches students to apply these strategies to other nonroutine situations.

The activities are organized by standard mathematics content strands for easy correlation with classroom texts. Each page is ready to be duplicated for student use. Corresponding notes for the teacher contain answers and helpful suggestions for classroom use. The activities are suitable for individuals, small groups, or the whole class and, with modifications, can be used over a wide range of ability levels. Most activities are self-explanatory; however, introductory discussion, modifications, or extensions may be appropriate, depending on the individual student's background.

Some teachers may find it suitable to work with the total class on the major question, then assign individuals or small groups to work the extensions in the looking-back section, since these extensions often apply and expand the work done in the main section. Many of the activities suggest working with a friend, since experience indicates that such interaction among students often enhances problem solving. Working together increases perseverance on the problems, and the students help each other discover patterns, meanings, and alternative ways to find solutions.

In addition to the Teacher's Notes corresponding to each activity, teachers will find the introduction helpful. It includes a discussion on the fundamentals of teaching problem solving and how the calculator can be an invaluable aid in this process. Recommendations in the spirit of George Polya, the great teacher of problem solving, are given, and the strategies used in the manual are explained.

Throughout, the intent of these activities is merely to suggest an approach to integrate teaching problem solving with calculators in the regular curriculum. It is hoped that these suggestions will be but a beginning—that teachers will go on to modify, adapt, and extend the ideas to fit their individual classrooms.

I am indebted to many for their help in preparing these materials, particularly to Jane Atwood, whose drawings inspired the illustrations, and to the many teachers and students who tried the activities and responded with invaluable comments.

INTRODUCTION

> Problem Solving: Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations.
>
> National Council of Supervisors of Mathematics, 1977
> *Position Paper on Basic Mathematical Skills*

> Problem solving [must] be the focus of school mathematics in the 1980s. . . . Mathematics programs [must] take full advantage of the power of calculators and computers at all grade levels.
>
> National Council of Teachers of Mathematics, 1980
> *An Agenda for Action: Recommendations for School Mathematics of the 1980s*

"Back to basics" is a currently popular slogan. However, as the statements above indicate, organizations concerned with mathematics education include more than computational skills in their lists of what constitutes the basics. Problem solving—the ability to reason and apply mathematics in problem situations—is one of the essentials they include. Such statements as the ones quoted here affirm that computational skills alone are not enough; they are important only as one set of tools needed to solve problems. In school mathematics a shift is needed from an almost exclusive emphasis on skills to a greater emphasis on the *processes* and *strategies* of problem solving. This is particularly true with the changing needs of a technological society. The increasing availability of calculators and computers for computation can facilitate such a shift, since they lessen the need for extensive emphasis on computational skills. Consequently, time must be spent on the systematic teaching of problem solving using nontextbook, nonroutine problems. Indeed, attention to the problem-solving *processes* is considered by some to be the single most important objective of mathematics instruction.

The calculator is a valuable aid for meeting this objective. It frees students from the burden of cumbersome computation and allows them to concentrate on the higher-level aspects of the problem situation. As calculators become as accessible as pencil and paper, their role in the classroom will expand. Properly used, the calculator does not become a crutch but rather a tool that increases the problem-solving efficiency of the user. In fact, many general problem-solving strategies are made more feasible and effective with calculators.

Instruction to Develop Problem-solving Skills

Expertise in problem solving does not develop incidentally as a result of working out the problems in standard texts. Textbook problems are often presented in groups, and the process or skill required to solve them is specific to that group. Doing such problems provides practice in using a particular skill on a particular type of problem—that is, in applying a mechanical process rather than giving students a true experience in problem solving. Thus, most texts need to be supplemented with activities that focus specifically on developing strategies and skills that are helpful in solving many types of problems. Educators have found that specific instruction in these strategies can improve the problem-solving ability of students.

This instruction must be more than a *descriptive* presentation of problems that can be successfully solved by the strategy. It must also be *prescriptive* in the sense that it indicates to the student when and

why and how to use the strategy over a wide variety of problems. It must focus on the process of obtaining the solution and on the application of the strategy, not just on the end product—the solution itself.

Such instruction is appropriate at all levels and with all students. In order to build their confidence, we must see that students experience success in their initial encounters with problem solving. We can accomplish this by carefully choosing problems of different degrees of difficulty according to the ability of the students. But when calculators are available, expect surprises. The calculator has been found to be a confidence builder, and students often undertake a wider range of problems when calculators are available to them. Many who previously had no interest in mathematics become intrigued and motivated to work out problems, and they become surprisingly successful. Sometimes it is the average student who shines at doing nonroutine problems with calculators.

The Two Components of Problem Solving

Problem solving can be considered as having two component parts. First, there is *exploration:* the discovery of possible relationships that might exist. Second, there is *confirmation:* the testing of these relationships. Exploration uses inductive processes and strategies that aid discovery. Confirmation uses deductive processes, including testing with examples and justifying the generalizations. The rigor of such justifications depends on the level of the student; some may be expected to do more formal proofs, and others should use their own words to explain why it is reasonable that the generalization works.

Students should be allowed to experience both these components of problem solving. They miss much of the excitement and satisfaction that comes from uncovering and proving ideas when school mathematics is limited to applying relationships that the teacher tells them about. A steady diet of routine problems or drill inhibits interest and hampers intellectual development. The teacher who challenges students with problems appropriate to their knowledge and helps them develop the skills to solve them gives those students a taste for, and a means of, independent thinking that can last a lifetime.

The Steps of Problem Solving

The famous teacher of problem solving, George Polya (*How to Solve It,* Garden City, N.Y.: Doubleday & Co., 1957), identified four basic steps of the problem-solving process. Successful problem solvers seem to proceed through these same steps in a wide variety of problem situations. Teaching students these steps gives them a structure to use to get started in the problem-solving process.

Step 1: *Understand the Problem*

Get to know the problem by checking the meaning of key words, determining relevant data, looking for relationships among the various parts of the data, identifying what is being asked. Often it is helpful for students to restate the problem in their own words.

Step 2: *Decide on a Plan*

Choose a plan, a method, a strategy to solve the problem. This is perhaps the most difficult part of the problem-solving process. Successful problem solvers consciously choose a plan, basing their choice on skills they have developed. Some develop these skills from previous experience working problems, but most students need specific instruction in various strategies. They need to be taught how to solve problems by organizing data in a chart, finding a pattern that fits their data, or making a guess and using the results to make a better guess. Experiences in using these strategies in problem situations can help students develop the skill to choose an appropriate strategy and apply it successfully to a problem.

Step 3: *Carry Out the Plan*

Carrying out the plan is often a routine, mechanical process requiring perseverance. The calculator is a suitable, beneficial tool at this point.

Step 4: *Look Back*

The importance of looking back must not be overlooked in the rush to be satisfied with the answer. It

is in this step, by reexamining and reconsidering the completed solution, that the learner more fully assesses and assimilates the solution process. Whenever possible, extend and generalize the solution to a class of similar situations by making up and exploring related problems. Discuss alternative methods for solving the problem; when students have practice in working out a problem in more than one way, it increases their understanding and builds their repertoire of problem-solving skills. Having students take time to reflect fully on their answers and on how they solved the problem will pay off the next time they have a similar problem to solve.

The Problem-solving Strategies

In this section, each strategy is discussed and illustrated with an example. These examples were chosen to help describe the strategy; alternative solutions to a particular problem may (and often do) exist and should be encouraged. The strategies are listed separately here for convenience, not to imply that they are distinct. Often one strategy leads to another or strategies are used in combination—such as making an organized list in order to find a pattern.

Strategy: *Find a Pattern*

The technique of identifying a pattern in given information or data, and then making a generalization that can be tested, is the basis of the inductive method used by scientists and mathematicians. The calculator is a valuable tool for this method, since it frees the student from the burden of computational drudgery where the underlying pattern may be lost in the lengthy steps of computation. The calculator only does the calculating; the student must do the observing, testing, and thinking to find the pattern. Exploring patterns, testing hunches, and arriving at generalizations is the heart of this strategy. Initially, students may need leading questions to help them become more observant of patterns.

Example (from Activity 5)

What happens when you multiply your favorite two-digit number by 101? Will this happen with any two-digit number? Why?

Discussion: This problem calls for dealing with a generalization—*any two-digit number*. Students need experiences with problems like this to learn how to use patterns to form generalizations when not all the cases can be examined. In this activity, they are led through the steps:

1. Collect data and examine it for a pattern.
2. Make a generalization ("What do you notice?").
3. Test the generalization on other cases.
4. Justify the generalization.

The rigor of the justification will depend on the level of the student. For younger students, the informal explanation using an example (as done in the activity) is appropriate. Others may do a more rigorous justification for the general case (see Teacher's Notes). At either level, explaining why a pattern works facilitates transfer to similar situations. In this case, recognizing how place value is involved leads to using 1001 as a factor to repeat the digits of a three-digit number.

Strategy: *Guess and Check*

Polya advocates, "Let's teach guessing." Once an initial guess is made, the student has a point of reference from which refinements can be made. A guess is a starting point; the problem is begun. Too many students (and teachers) have the attitude that it is wrong to make a guess, perhaps because the method is often called "trial and error," and errors are to be avoided. However, by checking a guess, one can learn how to make a better guess—a process that can be repeated until the solution is found. This is the powerful strategy of making *successive approximations*. With a calculator to check guesses, this is often a very efficient way to solve problems.

Example (from Activity 6)

What number when multiplied times itself results in 529?

Discussion: Any student can at least begin the problem by making a guess. Even a student with poor

estimation skills who might first guess 45 learns from the guess: checking shows that since $45 \times 45 = 2025$, this guess is much too large. The student now has information to use in making a better guess. By successively making better guesses (and sometimes using other information—for instance, "in order for the product to have a 9 in the ones place, the number must have a 3 or 7 in the ones place"), he or she can eventually find the solution.

Strategy: *Make an Organized List* or *Chart*

Making a systematic list of the known information is an organizational technique helpful to people in all walks of life. In science and mathematics, this method is routinely used as a means of identifying the given information and recording data in a manner suitable for examination. From such examinations, simple conclusions can often be reached. By recording or generating the data in a systematic way rather than haphazardly guessing, one is less likely to overlook data or cases that are part of the solution and less likely to repeat any—thus sparing many an "Oh, we already have that one!"

The basis of the organization depends on the type of data; it may be sequential by time (activities 7 and 19) or by relative size of numbers (activities 8B and 23). When several variables are involved that need to be compared, and when the values are changing while the process is being carried out, a chart format helps to identify all the possible combinations.

Example (from Activity 3)

Your team needs new baseball gloves and bats. If gloves cost \$15 and bats \$12, what equipment could you get for less than \$100?

Number of Bats

Number of Gloves

	0	1	2	3	4	5	6	7	8
0	\$0	\$12							
1	\$15								
2				\$66					
3									
4									
5									
6									
7									
8									

Discussion: Since three variables are involved (number of bats, number of gloves, and total cost) a chart is an appropriate way to organize the data. When the chart is filled in, all possible combinations of numbers that satisfy the conditions are evident. Although constructing and filling in a chart often reveals the solution to a problem, students should be asked to interpret their findings to ensure that the chart is meaningful to them.

Strategy Games

Well-chosen games can be an invaluable teaching aid. Games are intrinsically motivating, and this advantage, heightened when calculators are involved, can be used to give practice in problem-solving strategies. Games that are useful do not depend totally on chance but rather require the player to employ problem-solving skills in order to make wise moves and find winning strategies. If a game involves a trick, the player should be required to investigate the trick so as to be able to explain why it works and to know when and how to apply it to similar situations. Initially, carefully worded leading questions are needed to ensure that students go beyond random trial-and-error levels of game playing. In these ways, a game becomes a teaching device rather than simply an amusing pastime.

Example (from Activity 4)

The first player enters 1, 2, or 3 in the calculator. Then the players take turns adding 1, 2, or 3 to the sum on the display. The player who makes the addition which gives a display of 21 wins.

Discussion: After playing several times, students notice that there are subtargets, numbers which, if reached, ensure a win. By making an *organized list* of these subtargets (17, 13, 9, 5, 1), students may see a *number pattern* emerging (each is found by successively subtracting 4 beginning with the target number 21). Thus, by using problem-solving skills, the student has found the winning strategy (begin with 1, then on each turn reach the subtargets by adding in such a way that the sum of the two players' numbers is 4). A student who develops such a strategy knows why it works and can apply similar strategies to other games.

The problem-solving strategies developed in these activities are general strategies that can be applied to a wide range of problems. Once your students are familiar with these strategies, and having calculators to free them from the burden of tedious computations, they will find their own problems to investigate and explore. Be prepared to be surprised at the wide range of problems they will undertake with calculators when you encourage them to "try to find out for yourself!"

ACTIVITIES

Strand: Whole Number Addition
Activity 1A

Strategy: LOOK FOR A PATTERN

The Jackpot

Congratulations! You have just won the Can-You-Believe-It Jackpot! As winner, you will receive \$1 on the first day of the year, \$2 the second day, \$3 the third day, and so on, for a full year. (Unfortunately, it is not a leap year.)

Understand the problem

1. How much will your winnings be for the first 3 days? $1 + 2 + 3 =$ ______
   4 days? $1 + 2 + 3 + 4 =$ ______
   5 days? $1 + 2 + 3 + 4 + 5 =$ ______

Decide on a plan

2. What pattern do you notice? ____________________________________
   Hint: $1 + 2 + 3 + 4 = 10 = (4 \times 5) \div 2$
   5 is the sum of the first and last terms: $1 + 4 = 5$.
   4 is the number of terms being added.
   Fill in: $1 + 2 + 3 + 4 + 5 =$ (______ × ______) ÷ 2

3. Use your pattern to find the sum for the first 10 days. Then use your calculator to check your answer by adding all the numbers.
   $1 + 2 + 3 + \ldots + 9 + 10 =$ (______ × ______) ÷ ______ = ______
   (What do these dots mean?)

Carry out the plan

4. Use your pattern to find out the total amount of money you will receive from the jackpot for one year.
   ____

5. *Bonus:* How many days would it take you to receive a million dollars this way? ____________

Look back

6. Use the pattern to find these sums:
   $1 + 2 + 3 + \ldots + 100 =$ ________
   $1 + 2 + 3 + \ldots + 200 =$ ________
   $1 + 2 + 3 + \ldots + 300 =$ ________
   What do you notice? ____________________________________

7. Explain the pattern to a friend.

Looking Back: More on Summing Numbers

1. Use the pattern from activity 1A to find:
   1 + 2 + 3 + 4 + . . . + 99 + 100 = (______ × ______) ÷ 2 = ______

2. To find out why this pattern works, take a close look at what is happening. First, write the sum twice—forward and then backward, thus:

	1 +	2	+	3	+ . . . +	99	+ 100
	100 +	99	+	98	+ . . . +	2	+ 1
Add each pair down:	101 +	______	+	______	+ . . . +	______	+ 101

   How much is this sum? 100 sets of 101 = 100 × 101 = ________
   But this is twice the sum of the numbers 1 through 100. Why? ________________________
   So 1 + 2 + 3 + . . . + 99 + 100 = (100 × 101) ÷ 2 = ______

3. Now that you know why the pattern works, you can find out what other kinds of number sequences it will work on. Will the pattern work to find the sum of all the even numbers through 20?

   Complete the steps:

Write the sum twice:	2 +	4	+	______	+ . . . +	______	+ 20
	20 +	18	+	______	+ . . . +	______	+ 2
Add each pair down:	22 +	______	+	______	+ . . . +	______	+ 22

   How much is this sum? ______ sets of 22 = ______ × ______
   So 2 + 4 + 6 + . . . + 20 = (______ × ______) ÷ 2 = ______
   Check your answer by adding all the even numbers from 2 to 20 on your calculator.

4. Describe the pattern to a friend. Explain (a) how you find the numbers to multiply and (b) why you then divide by 2.

5. Use your calculator and the pattern you have discovered to find these sums:
   a) 3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 + 30 = (______ × ______) ÷ 2 = ______
   b) 4 + 14 + 24 + 34 + 44 + 54 + 64 + 74 + 84 + 94 = (______ × ______) ÷ 2 = ______
   c) 1 + 2 + 3 + . . . + 499 + 500 = (______ × ______) ÷ 2 = ______
   d) The sum of the odd numbers less than 100: ________
   e) The amount you would have if you saved \$1 the first week, \$2 the second week, \$3 the third week, and so on, for a year. ________________

What's the Number?

I'm thinking of a two-digit number. If the digits are added, their sum is 8. If the digits are reversed, the number formed is 18 less than my number. What is my number?

Understand the problem

1. Could my number be more than 99? ______ Less than 10? ______
   Why? ____

Decide on a plan

2. List all the pairs of digits that sum to 8. ______ ______ ______ ______

3. How can you find out which pair works?

   ____

Carry out the plan

4. Use the guess and check method to find which pair works.

Pair	Check	Does It Work?
1, 7	$71 - 17 = 54$	No: $54 \neq 18$
2, __		
3, __		
4, __		

What is the number? __________

Look back

Use the guess and check method to find these numbers:

5. The sum of the digits of a two-digit number is 11. If the digits are reversed, the number formed is 27 less than the original number. Find the original number. __________

6. The sum of the digits of a two-digit number is 5. When the digits are reversed, the number formed is 9 *more* than the original number. Find the original number. __________

Strand: Whole Number Addition
Activity 3

Strategy: MAKE A CHART

Ball Time

Your team needs new baseball gloves and bats. If gloves cost $15 and bats $12, what equipment could you get for less than $100?

Understand the problem

1. If one glove is $15, how much is the cost of two gloves? _______ Three gloves? _______ Four gloves? _______ If one bat is $12, how much is the cost of two bats? _______ Three bats? _______ Four bats? _______ What pattern do you notice? ______________________________

Decide on a plan

2. Use what you found in problem 1 to find the combined cost of 2 gloves and 3 bats. Is it less than $100? __________

3. To organize your work to find all possible orders that cost less than $100, make a chart. Explain the rows and columns of the chart; the cost for an order of 2 gloves and 3 bats is shown for you.

Number of Bats (columns); **Number of Gloves** (rows)

	0	1	2	3	4	5	6	7	8
0	$0	$12							
1	$15								
2				$66					
3									
4									
5									
6									
7									
8									

Carry out the plan

4. Fill in the chart, showing the total cost for different numbers of gloves and bats. How does the chart help you to answer the question? ______________________________

Look back

5. What patterns did you use to make completing the chart easy? What do you notice about the first row? About the first column? How do you find the entries as you go across a row? As you go down a column?

Target

With a friend and a calculator, play Target several times. Try to figure out how you can always be the winner.

Rules: Your "target" number—the number you are aiming for—is 21. The first player enters 1, 2, or 3 in the calculator. Then players take turns adding 1, 2, or 3 to the sum on the display. The player who makes the addition that gives a display of 21 wins.

After playing several times, answer these questions:

1. If you reach a display of 17, why do you know you can win on your next turn?

   ___

2. If you reach a display of 13, why do you know you can reach 17 on your next turn and so win the game?

   ___

3. We can call 17 and 13 "subtarget" numbers. What is the subtarget number you want to reach in order to be sure to reach 13 on your next turn?

   ___

4. What is the subtarget number you want to reach before 9?

   ___

5. List the subtarget numbers in order, beginning with the largest:

   ___

   How did you find them?

   ___

6. If you are the first player, explain how you could always be the winner.

   ___

7. Suppose the target number is 50, and the numbers to be added are 1, 2, 3, 4, or 5. What strategy would you use to always be the winner?

   ___

8. Make up another similar game and describe a strategy you could use so you would always be the winner.

   ___

Double Vision

What happens when you multiply your favorite two-digit number by 101? Will this happen with *any* two-digit number? Why?

Understand the problem

1. Which kind of number—47 or 147—should you choose to multiply by 101? Why?

____

Decide on a plan

2. Use your calculator to help you make a list of multiplication problems to explore:

101 × 47 = ______	101 × 62 = ______	101 × ______ = ______
101 × 23 = ______	101 × ______ = ______	101 × ______ = ______

Carry out the plan

3. Now study the products. Look for a pattern. What do you notice? ______________________

Look back

4. Why does this happen? *Hint:* Think of 101 as 100 + 1.
So 101 × 47 = (100 + 1) × 47 = (100 × 47) + (1 × 47) = 4700 + 47 = 4747.
What is going on here? ______________________

5. What should you multiply a three-digit number by in order to have the product repeat the digits? ____
Why? ______________________

Calculating

6. What calculator sequence can you use when you want to multiply several different two-digit numbers by 101 without keying 101 each time? ______________________

Strand: Whole Number Multiplication **Strategy: GUESS AND CHECK**
Activity 6 (Can be extended with Activity 14)

What's Your Guess?

What number when multiplied times itself results in 529?

Understand the problem

1. Here is a number sentence for the problem: $\square \times \square = 529$
   Should the same number go in both boxes? __________

Decide on a plan

2. Try a number. Multiply it times itself. Did it give a product of 529? Even if it wasn't the correct number, what did it tell you about the number that will work? ____________________________________

Carry out the plan

3. Use your calculator to try more guesses. Write down the results each time.

Guess	Check
15	$15 \times 15 = 225$
______	____________________
______	____________________
______	____________________

Look back

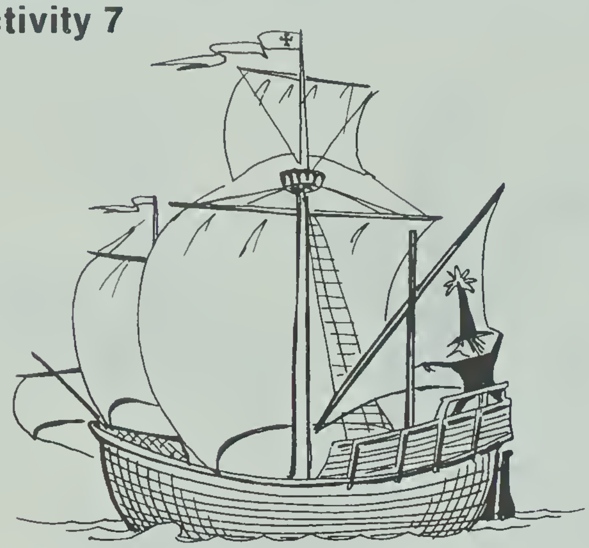
4. How did you use what you learned from checking a guess to make the next guess better? __________
   ____

5. Use the guess and check method to solve each of these:
   $\square \times \square = 2209$
   $\square \times \square \times \square = 704969$
   $\square \times \square \times \square \times \square = 1874161$

6. Do you think you could always find a whole number that would work? Try $\square \times \square = 110$.
   What do you find? ____

Calculating

7. What calculator sequence makes checking a guess easy?



Ancestors

How many direct ancestors from 19 generations ago do you have on your family tree? How many did you have at the time Columbus discovered America if each generation is about 30 years?

Understand the problem

1. Everyone is descended from two parents, four grandparents, eight great-grandparents, and so on. These people are your *direct* ancestors. Your parents are from one generation ago, your grandparents from two generations ago, and your great-grandparents from three generations ago. How many direct ancestors do you have on your family tree from three generations ago? ______ Four generations ago? ______

2. What year did Columbus discover America? ______
   How many years ago was that? ______
   How many generations ago? ______

Decide on a plan

3. Make a chart like this one to help you answer the problem:

Generations Ago	Number of Direct Ancestors
1	2
2	4
3	8
4	16
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
.	
.	
.	

Carry out the plan

4. Complete the chart. How many direct ancestors did you have when Columbus discovered America? ______

Look back

5. What pattern do you see in the numbers in the second column of the chart? (What calculator sequence helps you find this easily?) ______

6. How many direct ancestors did you have when the Pilgrims landed? ______

7. For further discussion: What was the world population when Columbus discovered America? If each person in your school had the number of ancestors you did at that time, what must be true about your ancestors?

______

Strand: Whole Number Multiplication
Activity 8A (Can be extended with Activities 8B and 24)

STRATEGY GAME

Multiplication Tic-Tac-Toe

Two players take turns. One chooses X and the other O. Then each picks two numbers from the factors list, multiplies them on the calculator, and puts his or her sign over that product on the grid. A player who gets a product that is already covered loses that turn. The first to get three in a row, column, or diagonal wins.

Game One

Factors: 7 12 19 26 35

133	910	494
312	84	228
245	420	665

Game Two

Factors: 8 13 29 31 46

1426	248	368
377	232	899
1334	104	403

Questions to think about while you play:

1. When there is a certain product you want to cover, how can looking at the ones digit help you? __________

2. If the product ends in a 0 or a 5, what do you know? __________

3. If the product is an even number, what do you know? __________

4. If the last pair you tried was too small, what do you know? __________

For these games, it takes *four* in a row, column, or diagonal to win:

Game Three

Factors: 4 11 17 24 31 35 43

341	68	1032	408
124	96	731	172
595	473	187	140
44	744	840	1333

Game Four

Factors: 5 14 19 29 33 41 57

1083	406	551	285
266	165	574	779
2337	145	1353	627
462	798	205	1881

Make Your Own Multiplication Tic-Tac-Toe Game

Understand the problem

1. For a three-in-a-row game, how many cells will there be in the board grid? ________

2. Will the cells be filled with *factors,* or with *products?* ____________

Decide on a plan

3. Choose the factors to use. You will need five factors.

4. Make an organized list of all the possible products for these factors. Example: For factors 3, 7, 16, 29, 31 the products are:

$3 \times 7 =$ ______	$7 \times 16 =$ ______	$16 \times 29 =$ ______	$29 \times 31 =$ ______
$3 \times 16 =$ ______	$7 \times 29 =$ ______	$16 \times 31 =$ ______	
$3 \times 29 =$ ______	$7 \times 31 =$ ______		
$3 \times 31 =$ ______			

Carry out the plan

5. Make a game board with nine of these products. Play the game.

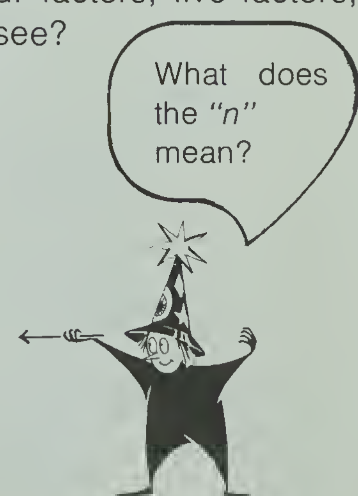
6. Start with your own list of five factors and make a game board.

Look back

7. How can you figure out the number of factors you will need to fill the 16 cells of a four-in-a-row game board grid?

   Use an organized list to list all the products for two factors, three factors, four factors, five factors, and so on. Then complete this chart with what you found. What pattern do you see?

Number of Factors	2	3	4	5	6		. . .	*n*
Number of Products	1	3		10			. . .	



Puzzling Squares

Think you know all about checkerboards? Then, how many squares are there on a checkerboard?

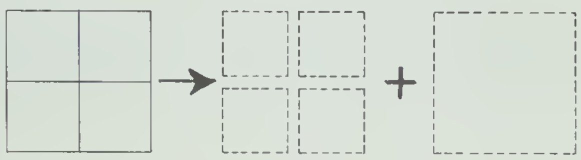
Understand the problem

1. A checkerboard has 8 small squares across the top and 8 down a side. How many of these small 1 × 1 squares are there altogether? ______

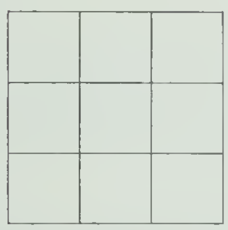
2. Draw a diagram of 4 of these smaller squares put together to form a larger square. How many of these 2 × 2 squares are there in a checkerboard? ______

Decide on a plan

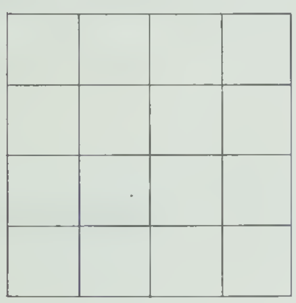
3. Starting with small checkerboards, count the squares and look for a pattern. How many squares?



→ 4 + 1 = 5



____ + ____ + ____ = 14



____ + ____ + ____ + ____ = ____

4. What kinds of numbers are being added? ________ Describe the pattern.

Carry out the plan

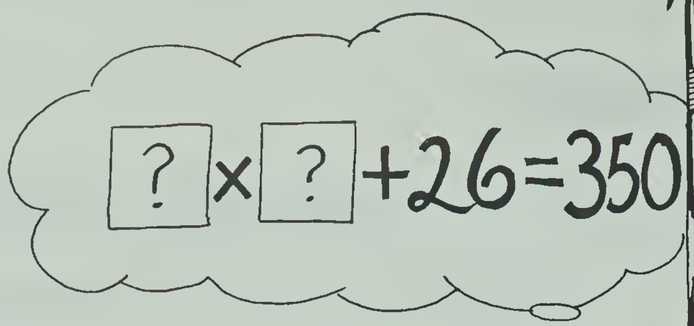
5. How many square numbers must be added for a regular 8 × 8 checkerboard? ________

6. Use your calculator to find the sum: ________

Look back

7. Describe how you would find the total number of squares in any sized checkerboard.

8. Make up a similar problem for a triangular playing board that is made up of smaller triangles.



What Is My Number?

I'm thinking of a number. If I multiply it times itself and then add 26, I get 350. What is my number?

Understand the problem

1. Write a number sentence that states the problem, using a box (□) to stand for the secret number.

____

Decide on a plan

2. Try a number: use your calculator to multiply the number times itself, then add 26.

   □ × □ = ______ + 26 = ______

3. Even if you didn't get 350, what did your guess and check tell you about the number you are looking for? ____

Carry out the plan

4. Use what you learned to make another guess. Try to find the number in as few guesses as you can.

Look back

5. How can recording your guesses and checks help you? ______________________

6. Use the guess and check method on these. Use as few guesses as you can.

   a) □ × □ − 19 = 150 *c)* (□ + 4) × 8 = 288

   b) □ × □ + □ = 600 *d)* (□ × □ ÷ 3) − 8 = 100

7. Challenge! Can you find two different whole numbers, one for □ and one for △, for each of these sentences?

   a) (□ + 2) × △ − 49 = 0 *b)* 2 × □ + 3 × △ = 100

Splash Time

Membership in the Splash Time Swim Club is $5.00. Each swim is $0.25 for members and $0.75 for nonmembers. How many times would you have to swim for it to be a better deal to be a member than a nonmember?

Understand the problem

1. If you are a member of the club and swim just one time, what does this swim cost? __________ If you are not a member, what does one swim cost? __________ What will two swims cost a member? __________ A nonmember? __________

Decide on a plan

2. What kind of chart helps you to organize your data?

Carry out the plan

3. Complete the chart that is started for you:

Number of Swims	1	2	3	4	5	6	7	8	9	10	11	12
Member	$5.25	$5.50				$6.50						
Nonmember	$0.75	$1.50				$4.50						

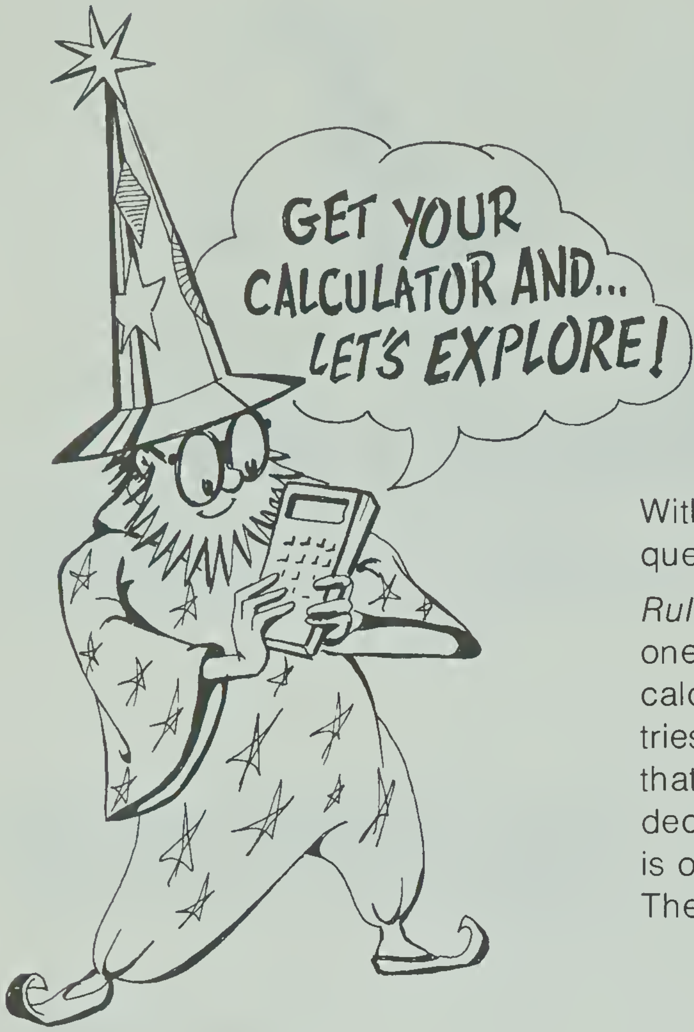
4. Use the chart to compare the actual cost per swim. What does the chart tell you: How many times must you swim for it to be a better deal to be a member? __________

Look back

5. Make a line graph to picture the information in the chart. Use one line to show the cost for members and another to show the cost for nonmembers. What does the crossing point of the lines mean?

Calculating

6. Describe a calculator sequence that can help you quickly find the entries for each row of the chart. (*Hint:* Use the constant feature of the calculator.)



Factors

With a friend and a calculator, play Factors. Then answer the questions below.

Rules: Start with a secret two-digit number and multiply it by one-digit numbers until you get a five-digit display on the calculator. Then pass the calculator to your friend. Your friend tries to factor the number, scoring one point for each division that gives a whole number. Your friend keeps dividing until a decimal number appears on the display; then his or her turn is over. Play four rounds, taking turns being the starting player. The player with the higher score wins.

After playing several rounds with these rules, allow two-digit factors to be used with the one-digit factors in generating the starting number.

As you play, find answers to these questions:

1. What kinds of factors should you use in generating the starting number in order to limit the score of your opponent as much as possible? ________
2. Why should the starting number be generated as described in the rules instead of simply keying in any five-digit number? ________
3. What kinds of two-digit starting numbers produce numbers that are the most difficult to factor?
____
4. If you recognize a large factor of the number on display, such as 100 being a factor of 72300, how should you play in order to make the largest possible score? ________

Strand: Decimal Numbers
Activity 13

Strategy: LOOK FOR A PATTERN

Nifty Nines

What happens when you divide a number by 9? By 99? By 999?

Understand the problem

1. Does the problem tell you what numbers to use as the divisor (the number to divide by)? ________
   Does it tell you what number or numbers to use as the dividend (the number to divide into)? ________

Decide on a plan

2. Try something. Pick any number that fits the problem and try it. Suppose you pick 3. Then find:

$3 \div 9 =$ ________

$3 \div 99 =$ ________

$3 \div 999 =$ ________

3. What seems to be happening? ________________________________

Carry out the plan

4. Look for a pattern by trying several other one-digit numbers and recording what you find:

______ ÷ 9 = ______	______ ÷ 9 = ______	______ ÷ 9 = ______
______ ÷ 99 = ______	______ ÷ 99 = ______	______ ÷ 99 = ______
______ ÷ 999 = ______	______ ÷ 999 = ______	______ ÷ 999 = ______

5. Use your pattern to predict what will happen when 7 is divided by 9, then 99, then 999. Now use your calculator to check your prediction. Were you right?

Look back

6. Will your pattern hold for two-digit numbers? Try it:

$23 \div 9 =$ ______	Now you pick a two-digit number to try:	______ ÷ 9 = ______
$23 \div 99 =$ ______		______ ÷ 99 = ______
$23 \div 999 =$ ______		______ ÷ 999 = ______

7. Go one more step: divide the two-digit numbers by 9999:

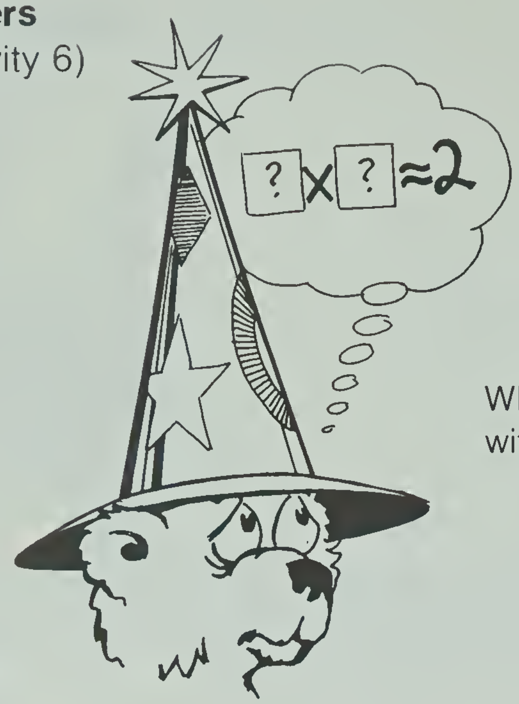
$23 \div 9999 =$ ______ ______ ÷ 9999 = ______

8. Describe the pattern to a friend. What do you think will happen with three-digit numbers? ________
____

9. The decimal numbers made by these divisions are called "repeating decimal numbers." Why do you think they are called this? ________________________________

Strand: Decimal Numbers
Activity 14 (Extends Activity 6)

Strategy: GUESS AND CHECK

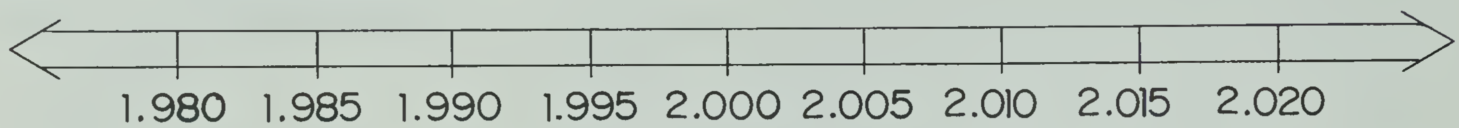


How Close Can You Get?

What number when multiplied times itself is within .01 of 2?

Understand the problem

1. Shade on the number line all the numbers that are within .01 of 2:



2. Since 1 times itself is 1, and 2 times itself is 4, what do you know about the size of the number that when multiplied times itself is close to 2? ____________________

Decide on a plan

3. Try a number between 1 and 2. Multiply it by itself. Is the answer within .01 of 2? If not, what does it tell you about the next number to try?

____

Carry out the plan

4. Use your calculator to keep trying numbers. Record your guesses on the chart below. Use the results of each check to make the next guess closer. The two guesses shown tell you your next guess must be between 1.4 and 1.5. What is a number between 1.4 and 1.5? (*Hint:* Remember 1.4 = 1.40 and 1.5 = 1.50.)

Guess	Guess times itself is:	Within .01 of 2?
1.5	1.5 × 1.5 = 2.25	2.25 - 2 = .25 > .01 No, too big
1.4	1.4 × 1.4 = 1.96	2 - 1.96 = .05 > .01 No, too small

Look back

5. Use the guess and check method to find:

a) ☐ × ☐ ≈ 17 (a number within .01) *b)* ☐ × ☐ ≈ 3 (a number within .001)

What Should You Buy?

You have $2.00. What two things could you buy to spend as much of the $2.00 as possible? (Assume you don't want more than one of any item; also assume there is no sales tax.)

Understand the problem

1. If you got a hot dog and a milk shake, how much would it cost? ________ How much change would you have left? ________

Decide on a plan

2. An organized list will help you consider every possibility. Complete the list:

Hamburger & Hot Dog	Hot Dog & Shake	Shake & Fries	Fries & Cola
Hamburger & Shake	Hot Dog & ________	Shake & ________	Fries & ________
Hamburger & ________	Hot Dog & ________	________ & ________	
Hamburger & ________	________ & ________		Cola & ________
Hamburger & ________			

Carry out the plan

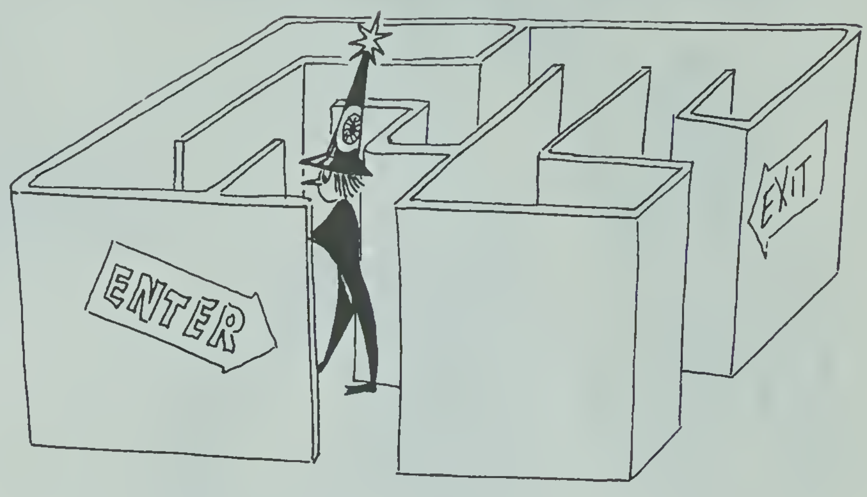
3. Use your calculator to find the cost for each combination on your list. What two things should you buy to spend as much of the $2.00 as possible? ________________

Look back

4. Which combinations could you eliminate just by estimating? ________________

5. If you could buy more than two items, how close could you come to spending $2.00? ________

Maze



With a friend, play Maze several times. Then answer the questions below.

Object: To achieve the lowest score by doing the operations for each line segment you trace to get from Start to Finish.

Materials: Playing board
Two calculators
One marker (a coin, square of cardboard, or button)

Rules: The game begins with the marker at Start and with both players entering the start number (100) in their calculators. Player 1 chooses a line segment connected with Start, moves the marker along it, and performs on his or her calculator the operation indicated. Player 2 continues by moving the marker along any connecting line segment and performing the operation indicated. The path may proceed in any direction. Segments may be used more than once but not on consecutive plays. When the marker reaches Finish, the player with the smaller calculator display wins.

Questions to answer after playing Maze:

1. If your choices on a play are × 1.02 or ÷ .5 or + 2, which should you choose? ______
   Why? ______
2. In general, what kinds of segment labels (operations) are helpful? ______
3. What kinds of segment labels (operations) are harmful? ______
4. How does your opponent's number affect your play?

   ______

Variations

1. The winner is the one with the larger number when Finish is reached.
2. A line segment may not be retraced. The game ends when Finish is reached or no move is possible.
3. Each player moves his or her own marker. The game ends when both have reached Finish.

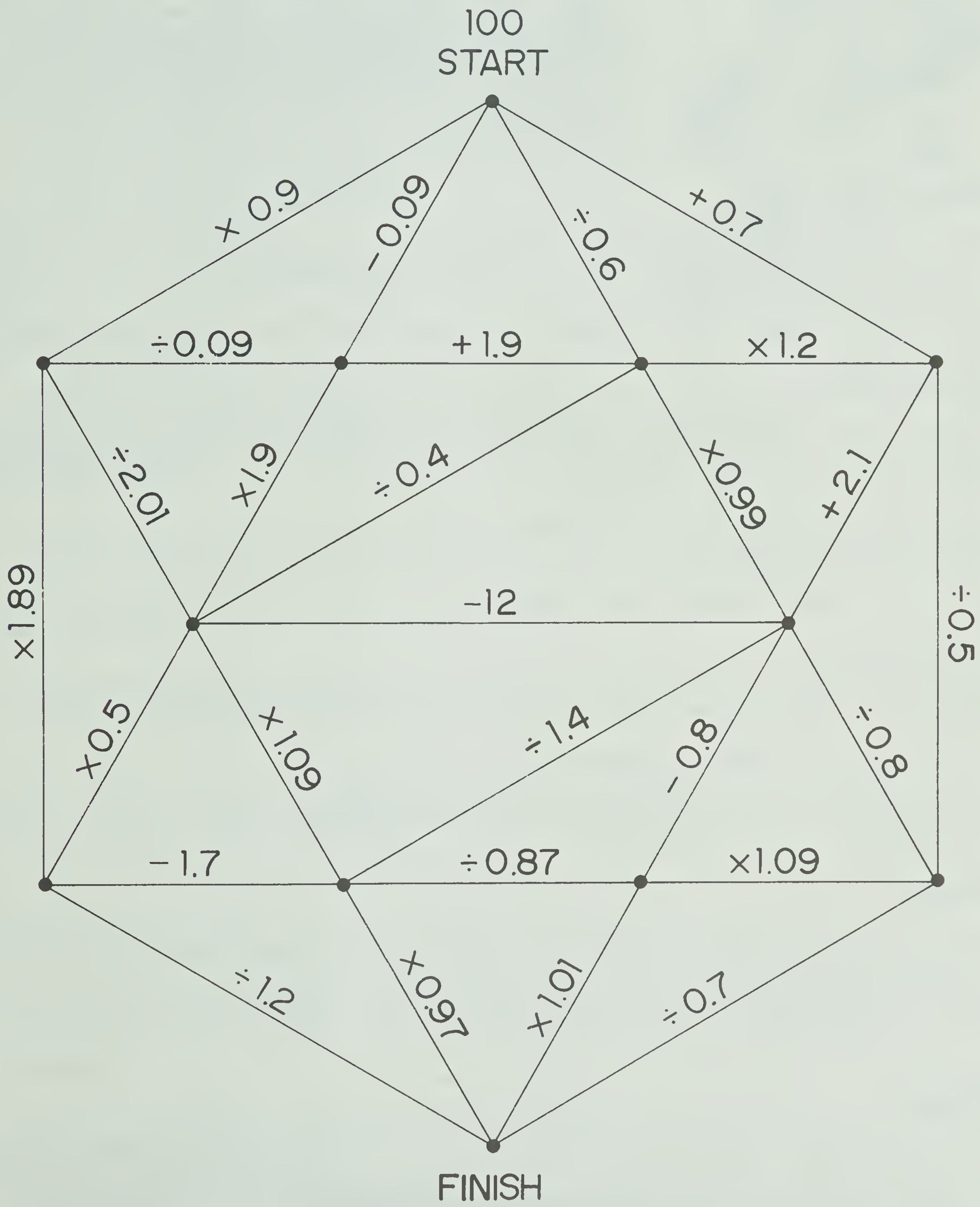
Extension

Now you design a maze for a game, and label the line segments. Be sure to consider how to place numbers with operation signs so that it is not too easy for your opponent to know which line segment to choose.

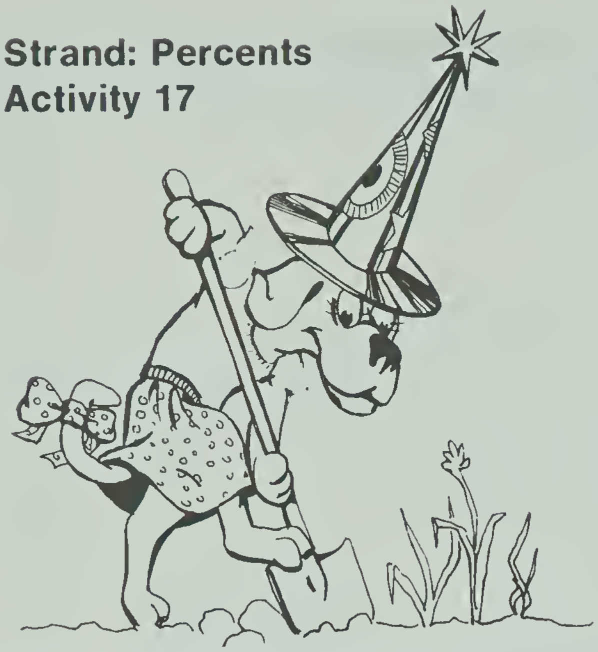
Activity 16

Maze

Playing Board



Strand: Percents
Activity 17



Strategy: LOOK FOR A PATTERN

Betty's Baffled

During March, Betty did not do her chores regularly, so her father said her April allowance would be cut 15%. During April, Betty worked hard to mend her ways. Her father, impressed, said her May allowance would be raised 15% over that of April. Is her May allowance the same as her March allowance?

Understand the problem

1. What was the first change made in Betty's allowance? __________ What was the second change? __________ How were they alike? __________ How were they different? __________

2. Do we know the amount of Betty's allowance? __________ Do we need to? __________

Decide on a plan

3. Explore. Pick an amount for Betty's March allowance and see if a 15% cut followed by a 15% raise gets you back to the original amount. For example, suppose Betty's March allowance was \$10. Use your calculator to find her April allowance:

   $\$10 - (\$10 \times 15\%) = \$10 -$ __________ $=$ __________

   What was her May allowance:

   $\$8.50 + (\$8.50 \times 15\%) = \$8.50 +$ __________ $=$ __________

Carry out the plan

4. What do you notice? __________

   Try a different amount. Does the pattern hold? __________

5. Does a 15% cut followed by a 15% raise ever get back to the original amount? __________ Why? __________

Look back

6. Use the method you have just used (looking for a pattern in sample cases) to answer these questions:
   a) What happens when a 15% raise is followed by a 15% cut? __________
   b) Is a 15% cut followed by a 10% raise the same thing as a 10% raise followed by a 15% cut? __________

Challenge!

   c) If you are allowed a 12% discount on a purchase but must pay a 6% sales tax, should the tax be computed before or after the discount is taken? __________ What percent of the original cost is the price you pay after both the discount and tax are computed? __________

Frustrating Foxy Fred

Foxy Fred has offered you a partnership in his business, Foxy Fred's Fast Fries and Floats. The offer is that you run FFFF&F but pay him a share based on how much the business makes. His share is to be a percentage equal to the number of hundreds of dollars of income. Should you accept Fred's offer? If you accept, what is the most you could keep after Fred takes his share?

Understand the problem

1. Suppose FFFF&F makes $1200, which is 12 hundreds. Then the rate for Fred's share is 12%. How much of the $1200 would Fred get? ________ How much would you keep? ________

Decide on a plan

2. Try another amount for the income. Use your calculator. Find the amount of Fred's share, and the amount you keep. From what you find, will your next guess for the income be larger or smaller in order for you to keep more? ________________

Carry out the plan

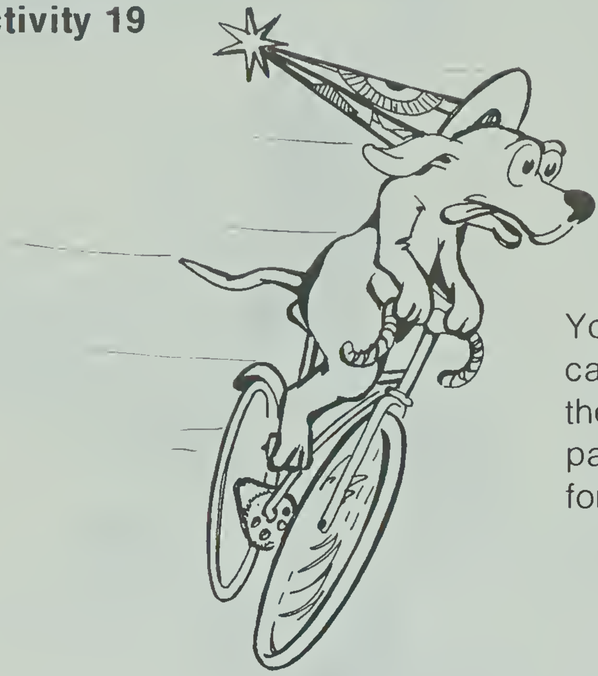
3. Continue making guesses at the income and finding how much you keep. Record your results. Be sure to use what you find from each guess to make the next guess better.

Income	Rate for Fred's Share	Amount of Fred's Share	Amount You Keep
$1200	12%	$144	$1056

Look back

4. What do you conclude? (Is Fred's offer a good arrangement for you? Who stands to benefit more, you or Fred?) ________________

5. What happens if the income is more than $10 000? ________________



Pedal Power

You have just purchased a bicycle for $160 using your credit card. You must pay interest at the rate of 1.5% per month on the unpaid balance. If you make a down payment of $20 and payments of $20 each month, how long will it take you to pay for the bicycle?

Understand the problem

1. After the down payment, how much is the unpaid balance? ________ How much is the interest on that balance for one month? ________ How much of the $20 payment goes toward paying off the balance? ________

Decide on a plan

2. How do you find the amount of the balance at the end of the first month? ________________
The end of the second month? ________________________________

Carry out the plan

3. Explain the first two rows of the chart below. Use your calculator. Continue to find the balance at the end of each month. Organize your work by completing the following list:

Month	Balance	Interest for One Month	Amount to Subtract to Obtain New Balance
0	$160.00 - 20.00 = $140.00	$140.00 × 1.5% = $2.10	$20.00 - 2.10 = $17.90
1	$140.00 - 17.90 = $122.10	$122.10 × 1.5% = $1.83	$20.00 - 1.83 = $18.17
2	$122.10 - 18.17 = ______	______ × 1.5% = ____	$20.00 - ____ = ____
3	______ - ____ = ______	______ × 1.5% = ____	$20.00 - ____ = ____
4	______ - ____ = ______	______ × 1.5% = ____	$20.00 - ____ = ____
5	______ - ____ = ______	______ × 1.5% = ____	$20.00 - ____ = ____
6	______ - ____ = ______	______ × 1.5% = ____	$20.00 - ____ = ____
7	______ - ____ = ______	______ × 1.5% = ____	$20.00 - ____ = ____

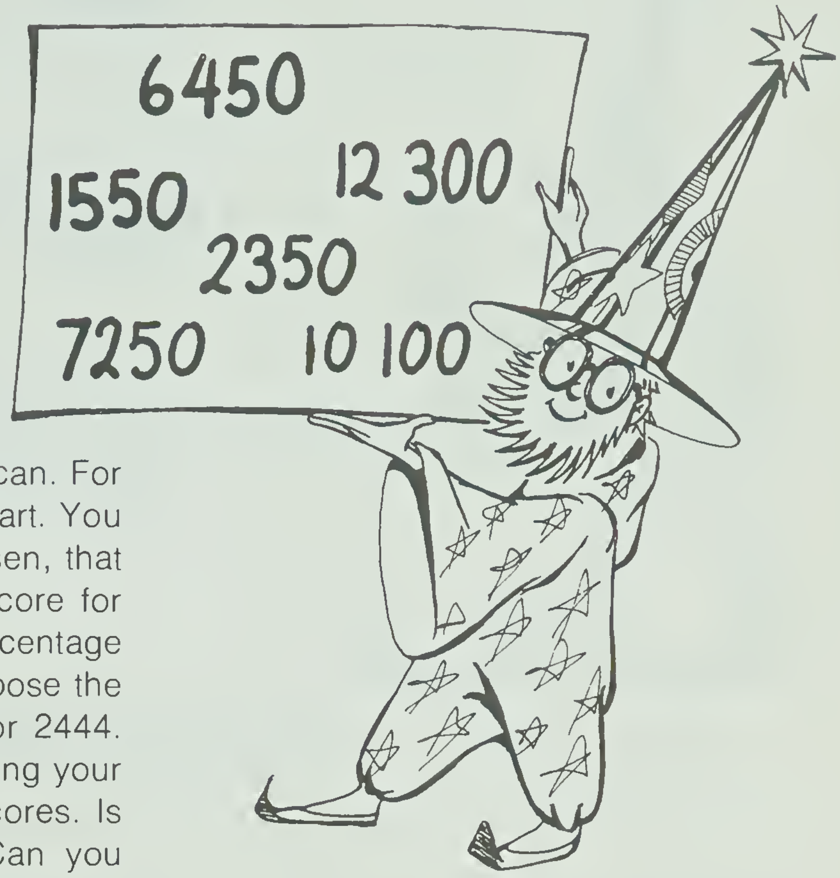
4. How many months will it take to pay for the bicycle? ________

Look back

5. What is the total amount of interest you paid? ________

6. How much would you save in interest if you paid $30 a month instead of $20 a month? ________

Solo Roll 'Em



Materials: 1 calculator
1 die

You have six turns to earn the highest total score you can. For each turn, roll the die, then circle a number on the chart. You may circle whichever number you like, but once chosen, that number may not be used again. To compute your score for that turn, increase the number circled by the percentage rolled on the die. For example, if you roll a 4 and choose the number 2350, your score is 2350 + (4% of 2350), or 2444. Continue rolling the die, choosing a number, and finding your score until all the numbers are used. Add all your scores. Is your score at least 42 000? Try the game again. Can you figure out how to get a higher score by choosing the numbers wisely?

After playing several times, answer these questions:

1. If you roll a 5 or 6 early in the game, which numbers should you choose from the chart? ________ ________ Why? ________
2. If you roll a 1 or 2 early in the game, which numbers should you choose from the chart? ________ ________ Why? ________
3. If two people play this game taking turns, how should the game strategy be different? (If you're not sure, try it with a friend and find out.) ________
4. What is the largest possible score after six turns? ________ What is the smallest possible score? ________ When would this happen? ________ What do you think is the most common score? ________ Why? ________

Strand: Geometry
Activity 21

Strategy: LOOK FOR A PATTERN

Circling In on Circles

For all circles, what can you say about the quotient of the circumference divided by the diameter?

Understand the problem

1. Find a circle. Point out its circumference. Then point out its diameter.
2. To find the quotient, what will you need to know about the circumference and diameter? ________

Decide on a plan

3. Since it is not possible to examine all circles, what would you know if you examined several circles and found a pattern? ________

Carry out the plan

4. Use a centimeter tape to measure the circumference and diameter of at least five different circles. Use the largest circles you can find. (Why?) Record the measures to the nearest centimeter in the chart. Then use your calculator to complete the chart.

Object	Measure of Circumference (C)	Measure of Diameter (d)	C÷ d

Look back

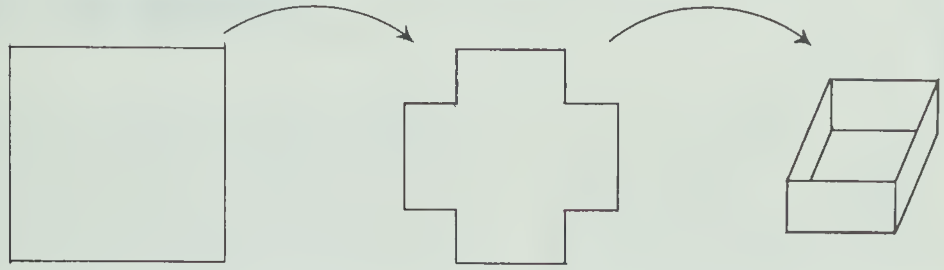
5. What do you notice? Do you think this is true for all circles? Why? ________
6. Look up the explanation of the Greek letter *pi* (π) in a dictionary or encyclopedia. How is it related to your investigation? ________

Strand: Geometry
Activity 22

Strategy: GUESS AND CHECK

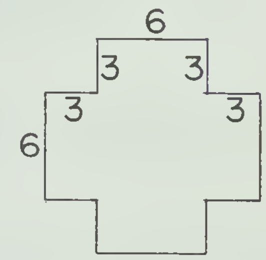
Go to the Corner!

An open box can be made by cutting equal corners from a square piece of cardboard:



If the cardboard is a 12-cm square, what size piece should be cut from the corner in order to make the largest possible box? (Use whole number measurements.)

Understand the problem



1. What is meant by "the largest possible box"?

2. Finish labeling the sides of the cardboard after 3-cm corners have been cut off. What are the length, width, and height of the box this would make? ____________ What would be the volume? ______

Decide on a plan

3. Decide on a different size for the corner, then use your calculator to find the volume that box would have. What do you notice? ____

Carry out the plan

4. Continue to guess a whole number size for the corner and check the volume of the box formed. Write down your results each time:

12-cm Square of Cardboard		
Guess for Size of Corner	Dimensions of Box Length × Width × Depth	Volume of Box
3	6 × 6 × 3	108

Look back

5. How do you know when you've found the largest possible box? ____________________________

6. *a)* What size corners should be cut from a 20-cm square of cardboard to make the largest possible box? ________________

   b) What size corners should be cut off a rectangular piece of cardboard of length 22 cm and width 15 cm to make the largest possible box? ________________

Fencing for Fowls

Farmer Greenfields has 36 meters of fencing. He wants to use it to make a rectangular chicken pen with the largest possible area. What should the dimensions of the pen be? (Use whole numbers.)

Understand the problem

1. If a rectangle has perimeter 36 m and width 5 m, what is the length? (Why must the sum of the length and width be 1/2 of 36, or 18?) What is the area of this rectangle? ______________

Decide on a plan

2. To consider every rectangle with perimeter 36, make an organized list:

Width	Length	Is Perimeter 36?	Area
1			
2			
3			
4			
5	13	Yes: $2 \times (5 + 13) = 36$	65
6			
7			
8			
9			

Carry out the plan

3. Find the area for each of the rectangles in your list. What should the dimensions of the pen be to give Farmer Greenfields the largest possible area? __________

4. Why does the list not need to extend beyond the width of 9? ______________

Look back

5. For a perimeter of 64, what are the dimensions of the rectangle with the largest area? __________ What do you notice?

Strand: Geometry
Activity 24 (Extends Activities 8A and 8B)

STRATEGY GAME

Multiplication Square-It

Object: Each player tries to complete squares by being the one to place a marker on the fourth corner of a square.

Rules: Two players take turns picking two numbers from the factors list, multiplying them with the calculator, and putting a marker over the product on the playing board. If the marker completes a square (covers the fourth corner of a square whose other three corners are already covered), the player scores. One point is scored for each square completed. If the product is already covered or is not one of the numbers on the board, the player does not put down a marker, and it is the next player's turn. The winner is the one with the highest score when all the numbers have been covered.

Factors: 7 13 18 27 38 41 43

301	1558	1634	1107
234	1026	189	1161
91	494	533	287
266	559	126	351

Questions to think about when you play:

1. When there is a certain product you want to cover, how can looking at the ones digit help you? ______________ How can estimation help? ______________
2. Did you find out how to score more than one point on a turn? ______________
   Remember: you get one point for each square you complete.
3. How many squares are there on the board? ________
4. How many possible products are not on the board? ________

Extension: Now you make up a game like this one and play it with a friend.

TEACHER'S NOTES AND SELECTED ANSWERS

Activity 1A *The Jackpot*

The sum of the first N counting numbers can be found by the formula $1 + 2 + 3 + \ldots + N = [N \times (N + 1)] \div 2$. For the students, the formula should result from finding a pattern in well-organized data.

4. The sum for 365 days is $(365 \times 366) \div 2 = \$66\,795$.

5. *Bonus:* The sum after 1414 days is $1 000 405, which is the first amount over a million dollars.

Activity 1B *Looking Back: More on Summing Numbers*

When students understand why a formula works, they not only are more likely to remember it but also will know when they can use it. Here the formula developed for finding the sum of N counting numbers is logically extended to find the sum of any *arithmetic sequence* (a sequence in which each term is equal to the preceding term and a constant). Expressing the formula in words may help: The sum is the first term plus the last term, times the number of terms, divided by 2.

5. *a)* $(33 \times 10) \div 2 = 165$; *b)* $(98 \times 10) \div 2 = 490$; *c)* $(501 \times 500) \div 2 = 125\,250$; *d)* $1 + 3 + 5 + \ldots + 99 = (100 \times 50) \div 2 = 2500$; *e)* $(52 \times 53) \div 2 = \$1378$.

Activity 2 *What's the Number?*

4. 53
5. 74
6. 23

Activity 3 *Ball Time*

A chart is used here to organize the data, since two variables (number of bats and number of gloves) need to be compared while values are changing. It is important that all students realize how the chart format helps identify all possible combinations of the variables. Some students may need help in constructing and reading the chart.

		Number of Bats								
		0	1	2	3	4	5	6	7	8
Number of Gloves	0	0	12	24	36	48	60	72	84	96
	1	15	27	39	51	63	75	87	99	111
	2	30	42	54	66	78	90	102	114	126
	3	45	57	69	81	93	105	117	129	141
	4	60	72	84	96	108	120	132	144	156
	5	75	87	99	111	123	135	147	159	171
	6	90	102	114	126	138	150	162	174	186
	7	105	117	129	141	153	165	177	189	201
	8	120	132	144	156	168	180	192	204	216

5. Be sure to discuss patterns in the chart and how using these patterns makes completing the chart easier. For example, since entries increase by 12 across a row, a possible key sequence for row 3 is 45 (+) 12 (=) (=) (=) (=). Similar patterns hold for the columns.

Activity 4 *Target*

This is a simple version of the classic game of nim. For a full discussion of the original version, see Martin Gardner, *The Scientific American Book of Mathematical Puzzles and Diversions* (New York: Simon & Schuster, 1959, pp. 151–161). After playing several times, most students will realize the game "is over" when 17 is reached. They should be encouraged, as in the questions, to find the pattern by making an organized list of the subtargets: 17, 13, 9, 5, 1. Noticing that the subtargets differ by 4 suggests the winning strategy: begin with 1, then on each turn reach a subtarget by adding the number that makes the sum of the two players' numbers 4. Be sure students apply this by extending and varying the problem, as in questions 7 and 8.

7. Begin with 2 (the remainder when 50 is divided by $5 + 1 = 6$), and on each turn add the number that makes the sum of the two players' numbers 6.
8. Possibilities: Let the winner be the one who does not reach the target number; start with 21 and subtract 1, 2, or 3.

Activity 5 *Double Vision*

5. Multiply by 1001 to have the three digits repeat in the product.
6. For preconstant calculators, key 101 (×) 47 (=) 23 (=) 62 (=), etc.

Activity 6 *What's Your Guess?*

You may want to introduce the square root key later, but do not use it for these. The method of successive approximation used here develops in the students an intuitive feel for roots and an idea of the relative size of the numbers involved. (See Activity 14 for extension.)

4. If the result is too large, try a smaller number. Also, since the ones digit must be a 9, only numbers with 3 or 7 in the ones place need to be tried.
5. $47 \times 47 = 2209$; $89 \times 89 \times 89 = 704969$; $37 \times 37 \times 37 \times 37 = 1874161$
6. Decimal number solutions are done in Activity 14.
7. *N* (×) (=) for any guess number *N*.

Activity 7 *Ancestors*

The sequence is the powers of 2 and is an example of a geometric sequence, since each term is equal to the previous term times a constant. The key sequence 2 (×) (=) (=) (=) . . . yields the powers of 2.

4. 1982 (or current year) - 1492 = 490 years, or 16.3 generations. $2^{16} = 65\,536$.
7. The numbers suggest we must have common ancestors.

Activity 8A *Multiplication Tic-Tac-Toe*

Although students may use random guessing at first, they should be encouraged after playing several times to use such strategies as those suggested in the questions. For an easier version, have the students write the factors below the product as they play. For extensions, see Glenda Lappan and Mary Jean Winter, "A Calculator Activity That Teaches Mathematics," *Arithmetic Teacher* 25 (April 1978): 21–23.

Activity 8B *Make Your Own Multiplication Tic-Tac-Toe Game*

Using an organized list avoids repetitions and omissions.

7. For four factors there are $1 + 2 + 3 = 6$ possibilities. For N factors there are $1 + 2 + 3 + \ldots + (n - 1)$ possibilities. Advanced students may find that the combinations formula for N things taken two at a time, $N!/2!\,(N - 2)!$, applies.

Activity 9 *Puzzling Squares*

Looking at similar, smaller cases often helps to find a pattern. In this activity, the smaller cases are investigated using counting techniques, then the pattern identified and used for the larger case where counting is not feasible.

4. The addends are the square numbers.
5. For an 8×8 board, eight square numbers must be added.
6. $64 + 49 + 36 + 25 + 16 + 9 + 4 + 1 = 204$.
7. For an $N \times N$ board:
   $N^2 + (N - 1)^2 + (N - 2)^2 + \ldots + 4 + 1 =$ number of squares.

Activity 10 *What Is My Number?*

5. Recording guesses and checks (see Activity 6) enables the problem solver to use information from previous guesses to make the next guess better.
6. 13; 24; 32; 18
7. *a)* $(5 + 2) \times 7 - 49 = 0$
   b) Challenge students to find as many solutions as possible. Note that △ must be an even number and less than 1/3 of 100. The sixteen solutions are:

□	2	5	8	11	14	17	20	23	26	29	32	35	38	41	44	47
△	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2

Activity 11 *Splash Time*

When entries increase by a constant amount, as in this chart, the calculator constant feature is helpful. To list the costs for members, use the key sequence 500 + 25 = = =, etc. To list the costs for nonmembers, use 75 + = = =, etc. In both sequences, the second addend is held constant by the calculator.

4. For ten swims, the total cost is the same for members and nonmembers, \$7.50. For more than ten swims, it is a better deal to be a member. (Be sure to have students interpret results after compiling a chart.)

Activity 12 *Factors*

This game encourages quick application of divisibility rules. Students tend to apply more sophisticated strategies after playing several rounds. Encourage this with questions such as those given.

1. Use numbers that are not easily recognized as factors, such as 7 or 13.
2. To make sure the starting number is not prime.
3. Large numbers that are not even (divisible by 2) or do not end in 5 (divisible by 5).
4. Use as many small factors as possible. Instead of dividing by 100 (for 1 point), divide by 2, then 2, then 5, then 5 (for 4 points).

Activity 13 *Nifty Nines*

Since a statement of what has happened in the general case ("all the time") is needed, an appropriate strategy is to examine specific cases that fit the conditions, then look for a pattern in these cases that holds in the general case. Informal explanations of why the pattern holds are usually sufficient; however, it is important that students know when the pattern found for one-digit numbers must be modified in order to hold for two-digit numbers.

5. $7 \div 9 = .\overline{7} = .777777$; $7 \div 99 = .\overline{07} = .070707$; $7 \div 999 = .\overline{007} = .007007$
6. For two-digit numbers, the pattern begins with two-digit divisors. (If the calculator rounds rather than truncates the last digit, answers will differ slightly.)
9. Nonending, or infinite *repeating decimals*.

Activity 14 *How Close Can You Get?*

This activity extends the concept of roots (see Activity 6) to those numbers that do not have whole number roots. These are examples of irrational numbers. They can be closely approximated but cannot be expressed exactly as a decimal number. For some students, you may want to introduce the square root notation in this activity. A variation of this activity is to play it as a game: the first to get a guess within the designated range wins.

4. $1.414^2 = 1.999396$; $2 - 1.999396 = .000604 < .01$
5. *a)* $4.123^2 = 16.999129$
   b) $1.732^2 = 2.999824$

Activity 15 *What Should You Buy?*

Students should note the pattern of organization used in the listing and discuss why this organized list avoids repeating or leaving out possible cases.

3.

1.29 + 1.05 = 2.34	1.05 + .65 = 1.70	.65 + .75 = 1.40	~~.75 + .49~~ ~~.49 + .55~~
1.29 + .65 = 1.94	1.05 + .75 = 1.80	.65 + .49 = 1.14	~~.75 + .55~~
1.29 + .75 = 2.04	1.05 + .49 = 1.54	.65 + .55 = 1.20	(eliminated
1.29 + .49 = 1.78	1.05 + .55 = 1.60		by estimation)
1.29 + .55 = 1.84			

A hamburger and shake would use as much of the $2.00 as possible.

Activity 16 *Maze*

As with most strategy games, the full benefit of Maze comes only after playing it for a while. It is then that students get above the trial-and-error level and begin to think how to apply strategies. The extension can give students valuable insights into the game.

Activity 17 *Betty's Baffled*

If students understand the process of working percent increase and percent decrease problems, they may want to use the appropriate calculator sequences to do these problems.

3. $8.50; $9.78
5. No; the bases involved are different. It is a common mistake to think than an *x*% decrease is exactly offset by an *x*% increase.
6. *a)* Since the bases are different, the result is a net loss.
   b) Yes.
   c) The order does not matter to the buyer; either way, the final cost is 88% of the original.

Activity 18 *Frustrating Foxy Fred*

3. Here are some possible answers for the chart:

Income	Rate	Fred's Share	Amount You Keep
$4900	49%	$2401	$2499
$5000	50%	$2500	$2500
$5100	51%	$2601	$2499

4. After the $5000 level, increases in income all go to Fred.

5. Technically, you start owing Fred money, since the rate is more than 100%.

Activity 19 *Pedal Power*

As a possible extension, discuss with students what they observe about the "cost" of buying things on time (that is, how the amount paid in interest adds up).

3.

Month	Balance	Interest	Amount to Subtract
0	$140.00	$2.10	$17.90
1	122.10	1.83	18.17
2	103.93	1.56	18.44
3	85.49	1.28	18.72
4	66.77	1.00	19.00
5	47.77	.72	19.28
6	28.49	.43	19.57
7	8.92	.13	19.87

4. In month 8, a payment of $9.05 pays off the bike.

5. Total interest paid is $9.05 (a coincidence that the two answers are the same).

Activity 20 *Solo Roll 'Em*

After playing awhile, students should begin to develop strategies for making wise choices. For example, if a 6 comes up, and the largest number is still available, it should definitely be played. If a 5 comes up early in the game, a decision must be made whether to wait to use the largest number on the chance of getting a 6. They should use 1's and 2's with the smaller numbers. Students should discuss why the outcome of the game is uncertain: you can't be sure each number of the die will come up.

Activity 21 *Circling In on Circles*

By choosing their own circles to investigate, students become convinced that the result holds for all circles, not just teacher-chosen ones. Using large circles makes errors in measurements relatively smaller. Discuss why the results are not exactly π (because no measurement is exact). Have students list possible sources of inaccuracy or lack of precision.

5. Numbers in the last column are close to 3.14, which is an approximation for π, defined to be the ratio of the circumference divided by the diameter for all circles.

Activity 22 *Go to the Corner!*

For some students, making a physical model will help them to visualize the problem. This problem is a standard calculus problem that is solved using derivatives. Students will be interested to learn that they know how to solve a calculus problem.

4. Size of Corner — Dimensions — Volume

Size of Corner	Dimensions	Volume
1	$10 \times 10 \times 1$	100
2	$8 \times 8 \times 2$	128
3	$6 \times 6 \times 3$	108
4	$4 \times 4 \times 4$	64
5	$2 \times 2 \times 5$	20

5. The largest is $8 \times 8 \times 2$, since the volumes then begin decreasing as the size of the corners increases.

6. *a)* For a 20-cm square, the largest box is $14 \times 14 \times 3$, with a volume of 588 cm^3.
   b) For a 22×15 rectangle, 3-cm corners produce the largest box: $9 \times 16 \times 3$, with a volume of 432 cm^3.

Activity 23 ***Fencing for Fowls***

Have students review the distinction between perimeter and area.

3. The largest area is that of a square with the given perimeter—in this case, 9×9.

5. Again, the largest area results from using a square, 16×16. (In general, the area increases as the number of equal-sized sides increases. Thus, a circle—the shape approached as the number of equal-sized increased—has the largest area for a fixed perimeter. This raises the question: Why don't farmers plant circular fields?)

Activity 24 ***Multiplication Square-It***

1. If the ones digit of the product is odd, the two factors must both be odd.

2. Because of overlapping squares, a number may be the corner of more than one square.

3. Part of the intrigue of this game is noticing squares that were overlooked at first, particularly the tilted squares.

4. With seven factors, there are $6 + 5 + 4 + 3 + 2 + 1 = 21$ products (see Activity 8), and so five were not used.

DATE DUE SLIP

NOV 18 '91	RETURN SEP 26 '96
DEC 03 '91	RETURN OCT 15 '96
DEC 14 '91	RETURN FEB 12 97
DEC 18 1991 RETURN	
EDUC FEB 22 '92	RETURN MAY 2
	RETURN JUL 03 '97
FEB 23 1992 RETURN	
FEB 1 '93	
EDUC MAR 04 '93	
MAR 04 RETURN	
DUE EDUC JUL 22 '93	
1993 JUL 21 RETURN	
RETURN SEP 06 '94	
RETURN NOV 03 '94	

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